OKA-CARTAN FUNDAMENTAL THEOREM ON STEIN MANIFOLDS

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CONTENTS

INTRODUCTION

The aim of this report is to prove the *Oka-Cartan fundamental theorem*, also known as *Cartan theorem B*. The proof will closely follow the exposition in [Nog16].

Theorem. *Let M be a Stein manifold, and let $\mathcal{F}$ be a coherent sheaf over M. Then we have*

$$H^q(M, \mathcal{F}) = 0 \quad q \geq 1$$

The result relies on the proof of equivalent statements for manifolds of increasing complexity:

- convex cylinder domains (step 2-3);
- analytic polyhedra (step 4);
- holomorphically convex domains (step 5);
- Stein manifolds (section 3);

As a corollary, in the last section we will prove

Theorem (Cartan theorem A)**.** *Let M be a Stein manifold and $\mathcal{F}$ a coherent sheaf over M. Then $\mathcal{F}$ is spanned by finitely many global sections, i.e., for a suitable $N \in \mathcal{N}$, the following sequence is exact:*

$$\mathcal{O}_M^N \longrightarrow \mathcal{F} \longrightarrow 0$$

Theorem (Analytic de Rham theorem)**.** *Let M be a Stein manifold of dimension n. Then it holds that*

$$H^q(M, \mathbb{C}) \cong H^q_{AdR}(M, \mathbb{C}) \quad q \geq 0$$

In particular, $H^q(M, \mathbb{C}) = 0$ when $q > n$.

On a general manifold the last result may fail as shown in subsection 1.3.

The Oka-Cartan fundamental theorem solves the first and the second Cousin problem and it plays a key role in the GAGA correspondence.

1 PREPARATION

In the sequel, we will use the following conventions:

- $\mathcal{O}_M$ is the sheaf of holomorphic functions over a complex manifold M;
- $B(a,r) = \{z \in \mathbb{C}^n \mid |z-a| < r\}$ is the ball centered at a of radius r;
- $D(a,r) = \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid |z_i - a_i| < r_i, i = 1,\dots,n\}$ is the polydisk centered at $a = (a_1,\dots,a_n)$ of radius $r = (r_1,\dots,r_n)$.
- Let $f\colon M \to N$ be a holomorphic map between complex manifolds. The functor f^* is the *inverse image* functor in the category of $\mathcal{O}_M$-modules and f_* is the *direct image* functor in the category of $\mathcal{O}_N$-modules. More precisely, given a $\mathcal{O}_M$-module $\mathcal{F}$ and a $\mathcal{O}_N$-module $\mathcal{G}$,

$$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_N} \mathcal{O}_M$$
$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \quad \forall U \text{ open in } N$$

  (f^*, f_*) is an *adjoint situation*, or f^* is left adjoint to f_*. For further details, the reader is referred to [Dem12].

1.1 Coherent sheaves

In this subsection we recall some properties of coherent sheaves. For further details, the reader is referred to [Nog16; Dem12].

Theorem 1 (Serre theorem)**.** *Let be given the short exact sequence of $\mathcal{O}_M$-modules*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

If two of the sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are coherent, then all three are coherent.

Theorem 2. *Let $F, \mathcal{G}$ be coherent $\mathcal{O}_M$-modules over M. Then the tensor product*

$$\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{G}$$

is a coherent sheaf.

Let X be a closed complex submanifold of a complex manifold M. We denote $i\colon X \hookrightarrow M$ the inclusion map.

Definition 3 (Geometric ideal sheaf)**.** The *geometric ideal sheaf* is the kernel of the surjection $\mathcal{O}_M \to i_*\mathcal{O}_M$, defined by

$$\mathcal{O}_M(U) \to i_*\mathcal{O}_M(U) \cong \mathcal{O}_X(U \cap X) \quad \forall U \text{ open in } M$$
$$f \mapsto f|_X$$

Definition 4 (Simple extension of a sheaf)**.** The *simple extension* of a sheaf $\mathcal{F}$ over X is the direct image of $\mathcal{F}$ with respect to the inclusion map $X \hookrightarrow M$. It is denoted $\widehat{\mathcal{F}} = i_*\mathcal{F}$.

If $X = \{a\}$, an $\mathcal{O}_X$-module over X is just a module over the ring $\mathbb{C}$ and its simple extension is the skyscraper sheaf of that $\mathbb{C}$-module centered in a. Furthermore, any $\mathcal{O}_M$-module $\mathcal{F}$ supported on the point a is a skyscraper sheaf $\widehat{\mathcal{F}_a}$ centered in a. Indeed, sections of $\mathcal{F}$ over a neighbourhood of a are determined by their projection in the stalk $\mathcal{F}_a$ of a, while they are identically zero over an open subset which does not contain a.

Theorem 5. *Let X be a closed complex submanifold of a complex manifold M. Then the following propositions hold:*

- *the geometric ideal sheaf $\mathcal{I}_X$ of X is a coherent sheaf over M;*
- *the simple extension extension $\widehat{\mathcal{O}_X}$ of the sheaf $\mathcal{O}_X$ over M is coherent over M;*
- *the simple extension extension $\widehat{\mathcal{F}}$ of a coherent sheaf $\mathcal{F}$ over M is coherent over M.*

With the notation above,

Proposition 6. $H^q(X,\mathcal{F}) \cong H^q(M,\widehat{\mathcal{F}}) \quad q \geq 0$

Proof. Choose $\mathcal{U}$ a covering of M.

$$\begin{aligned} \mathcal{C}^q(\mathcal{U}, i_*\mathcal{F}) &= \prod_{(j_0,\dots,j_q)\in A^{q+1}} i_*\mathcal{F}(U_{j_0} \cap \dots \cap U_{j_q}) \\ &= \prod_{(j_0,\dots,j_q)\in A^{q+1}} \mathcal{F}(i^{-1}(U_{j_0}) \cap \dots \cap i^{-1}(U_{j_q})) \\ &= \mathcal{C}^q(i^{-1}(\mathcal{U}), \mathcal{F}) \end{aligned}$$

The inverse image $i^{-1}(\mathcal{U})$ is a covering of X, and since X is closed, any covering of X can be considered as the restriction of a covering of M. Hence,

$$H^q(X,\mathcal{F}) \cong H^q(M, i_*\mathcal{F}) \quad q \geq 0$$

since they are colimits of the same diagram. □

Let $f\colon M \to N$ be a biholomorphic map between complex manifolds.

Proposition 7. $H^q(M,\mathcal{F}) \cong H^q(N, f_*\widehat{\mathcal{F}}) \quad q \geq 0$

Proof. Choose $\mathcal{V}$ a covering of N. As in proposition 6,

$$\mathcal{C}^q(\mathcal{V}, f_*\mathcal{F}) = \mathcal{C}^q(f^{-1}(\mathcal{V}), \mathcal{F})$$

The diagram of the coverings of N is isomorphic to the diagram of the coverings of M via the inverse image of f. Hence,

$$H^q(M,\mathcal{F}) \cong H^q(N, f_*\mathcal{F}) \quad q \geq 0$$

□

In general, let $f\colon M \to N$ be a holomorphic map between complex manifolds. The right exactness of the inverse image functor f^* implies the coherence of the sheaf $f^*\mathcal{G}$, where $\mathcal{G}$ is any coherent sheaf over N. If f is a biholomorphism, also the direct image functor f_* is right exact, hence $f_*\mathcal{F}$ is coherent for any coherent sheaf $\mathcal{F}$ over M. By proposition 7,

$$H^q(M,\mathcal{F}) \cong H^q(N, f_*\mathcal{F}) \quad q \geq 0$$

Therefore, in the case of biholomorphic manifolds M and N, we deduce the following theorem.

Theorem 8. *Suppose that $H^q(N,\mathcal{G}) = 0$ for every coherent sheaf $\mathcal{G}$ over N. Then $H^q(M,\mathcal{F}) = 0$ for every coherent sheaf F over M.*

1.2 Holomorphic convexity

Our purpose is to prove the vanishing of some cohomology groups. It is natural to deal with manifolds satisfying a property

1. invariant under biholomorphic maps;
2. stable under finite intersections.

We require 1 as the vanishing property is also invariant under biholomorphisms, and we require 2 because we need Leray coverings to work with Čech cohomology. For the same reason, we will mainly deal with polydisks.

We will now introduce the basic notion of holomorphic convexity. For further details, the reader is referred to [Nog16; Dem12].

Definition 9 (Holomorphic convex hull)**.** Let M be a complex manifold and $K \subseteq M$ be a compact subset. The *holomorphic convex hull* of K is the set

$$\widehat{K} = \widehat{K}_{\mathcal{O}(M)} = \{z \in M \mid |f(z)| \leq \sup_K |f(z)| \,\forall f \in \mathcal{O}(M)\}$$

We list some elementary properties of the holomorphic convex hull.

1. $\widehat{K}$ is a closed set containing K;
2. $\widehat{\widehat{K}} = \widehat{K}$;
3. $\widehat{K}$ contains all the relatively compact connected components of $M \smallsetminus K$;
4. if Ω is a domain in $\mathbb{C}^n$, then $\widehat{K}_{\mathcal{O}(\Omega)}$ is contained in the convex hull of K;
5. if Ω_1, Ω_2 are domains in $\mathbb{C}^n$, $\Omega_1 \subseteq \Omega_2$, then $\widehat{K}_{\mathcal{O}(\Omega_1)} \subseteq \widehat{K}_{\mathcal{O}(\Omega_2)}$.

Since the convex hull of a compact set K in $\mathbb{C}^n$ is still compact, $\widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ is compact. When Ω is arbitrary, $\widehat{K}_{\mathcal{O}(\Omega)}$ is not always compact. For example, if $\Omega = \mathbb{C}^2 \smallsetminus \{0\}$, by Hartogs theorem $\mathcal{O}(\Omega) = \mathcal{O}(\mathbb{C}^2)$, hence the holomorphic hull of the sphere S^3 is the non compact set $\widehat{S^3} = \overline{B}(0,1) \smallsetminus \{0\}$, as it is easily seen applying properties 1, 3 and 4 above.

Definition 10 (Holomorphic convexity)**.** A complex manifold M is said to be *holomorphically convex* if the holomorphic hull $\widehat{K}_{\mathcal{O}(M)}$ of every compact subset $K \subseteq M$ is compact.

Examples of holomorphically convex domains include:

- Any domain in $\mathbb{C}$, as a consequence of Runge theorem.
- Any convex domain in $\mathbb{C}^n$, by property 3 above.
- A *convex cylinder domain* in $\mathbb{C}^n$ (i.e., a product of n convex domains in $\mathbb{C}$), since it is biholomorphic to a polydisk. More generally, it is sufficient to take a product of simply connected domains, thanks to the Riemann mapping theorem.

On the other hand, for instance $\mathbb{C}^2 \smallsetminus \{0\}$ is not holomorphically convex for the aforementioned reasons. Using definition 10 and property 5, the following proposition is easily proved.

Proposition 11. *Holomorphic convexity is invariant under biholomorphic mappings, and stable under finite intersections.*

1.3 Sheaf resolutions

By Poincaré lemma, $\bar{\partial}$-Poincaré lemma, and holomorphic Poincaré lemma, the following sequences are exact:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}^0_M \xrightarrow{d} \mathcal{E}^1_M \longrightarrow \dots$$

$$0 \longrightarrow \mathcal{O}^p_M \longrightarrow \mathcal{E}^{p,0}_M \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1}_M \longrightarrow \dots$$

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_M = \mathcal{O}^{(0)}_M \xrightarrow{\partial} \mathcal{O}^{(1)}_M \longrightarrow \dots$$

where $\mathcal{E}^p, \mathcal{E}^{p,q}, \mathcal{O}_M^{(p)}$ denote respectively the sheaf of smooth p-forms, (p,q)-forms and holomorphic p-forms. Since the first two resolutions are acyclic, the abstract de Rham theorem implies

$$\begin{aligned} H^q_{dR}(M, \mathbb{R}) &\cong H^q(M, \mathbb{R}) && q \geq 0 \\ H^{p,q}_{\bar\partial}(M, \mathbb{C}) &\cong H^q(M, \mathcal{O}_M^{(p)}) && q \geq 0 \end{aligned}$$

In particular, for $p = 1$, the second resolution gives

$$H^{0,q}_{\bar\partial}(M, \mathbb{C}) \cong H^q(M, \mathcal{O}_M) \quad q \geq 0$$

By $\bar\partial$-Poincaré lemma ([GH78, p.25]), if M is a polydisk (or any convex cylinder domain), we conclude

$$H^q(M, \mathcal{O}_M) = 0 \quad q \geq 1$$

This is a particular case of the Oka-Cartan fundamental theorem.

Conversely, note that in general the third resolution above is *not* acyclic, and $H^q_{AdR}(M, \mathbb{R}) \ncong H^q(M, \mathbb{C})$. For example, let M be $\mathbb{C}^2 \smallsetminus \{0\}$. Taking the real tensor product of the first resolution with the constant sheaf $\mathbb{C}$, we obtain

$$H^q_{dR}(M, \mathbb{C}) \cong H^q(M, \mathbb{C}) \quad q \geq 0$$

Since M is homotopy equivalent to the sphere S^3, we obtain $H^3_{dR}(S^3) = \mathbb{C}$. But clearly, $H^3_{AdR}(M, \mathbb{C}) = 0$, because M has complex dimension 2. We will prove in the last section, as a corollary of the Oka-Cartan fundamental theorem, that the analytic de Rham cohomology of Stein manifolds is isomorphic to the cohomology of the constant sheaf $\mathbb{C}$.

2 THE FUNDAMENTAL THEOREM

Theorem 12 (Oka-Cartan fundamental theorem). *Let $\Omega \subseteq \mathbb{C}^n$ be a holomorphically convex domain, and let $\mathcal{F}$ be a coherent sheaf over Ω. Then we have*

$$H^q(\Omega, \mathcal{F}) = 0 \quad q \geq 1$$

The proof is rather long, hence it is divided in several steps for ease of exposition.

STEP 1. *Our first claim is: for an arbitrary sheaf $\mathcal{F}$ over any domain Ω, it holds that $H^q(\Omega, \mathcal{F}) = 0$, when $q \geq 2^{2n}$.*

Firstly, we choose a covering of Ω made of compact cubes $\{E_\alpha\}_{\alpha \in A}$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Then we choose another covering $\{U_\alpha\}_{\alpha \in A}$ of relatively compact open cubes such that $E_\alpha \Subset U_\alpha \Subset \Omega$. Since U_α is relatively compact, it intersects a finite number of cubes of the first covering, hence without loss of generality we can suppose that E_α is the only cube completely contained in U_α.

Note that the neighbourhood of a vertex has non-empty intersection with 2^{2n} cubes (of the first covering) at maximum. Therefore, intersecting more than 2^{2n} distinct cubes yields the empty set. Given a cochain $(f_{i_0,\ldots,i_q})$ in $\mathcal{C}^q(\{U_\alpha\}, \mathcal{F}) = \prod_{(i_0,\ldots,i_q) \in A^{q+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q})$, the cocyle condition implies that, whenever two indices in $\{i_0, \ldots, i_q\}$ coincide, we must have $f_{i_0,\ldots,i_q} = 0$. We conclude a q-cocycle is uniquely determined by sections over intersection of $q+1$ distinct open sets. This remark proves the claim.

STEP 2. Let Ω be a convex cylinder domain of complex dimension n. By 8, thanks to the Riemann Mapping Theorem, we can assume that Ω is an open cube. Let $\{\Omega_\nu\}$ be an exhaustion[1] by relatively compact open cubes in Ω.

1 Let Ω be a manifold. An exhaustion of Ω is a collection of open subsets $\{P_\nu\}$ such that $P_\nu \Subset P_{\nu+1}$ and $\Omega = \bigcup P_\nu$.

We want to prove that for any coherent sheaf $\mathcal{F}$ over Ω, $H^q(\Omega_\nu, \mathcal{F}) = 0$, for $q \geq 1$.

We denote by $\{E_{\nu\mu}\}$ a grid of finite compact cubes whose union is $\overline{\Omega}_\nu$. We require the $E_{\nu\mu}$ to be so small that each of them is contained in an open set $U_{\nu\mu}$ with the following properties:

- $E_{\nu\mu} \Subset U_{\nu\mu}$;
- for a suitable $N_{\nu\mu} \in \mathcal{N}$, there exists an exact sequence

$$\mathcal{O}_{U_{\nu\mu}}^{N_{\nu\mu}} \xrightarrow{\varphi_{\nu\mu}} \mathcal{F}|_{U_{\nu\mu}} \longrightarrow 0$$

The existence of such $U_{\nu\mu}$ is guaranteed by the following reasoning: we apply the definition of coherence to find open sets around each point in Ω_ν, and then we use Lebesgue number lemma to obtain a number δ, which we choose to be the diameter of the cubes forming the grid. Applying Cartan merging lemma ([Nog16, lemma 4.2.17]), a finite generator system is found by glueing together those for $\mathcal{F}(U_{\nu\mu})$ and $\mathcal{F}(U_{\nu\mu'})$ (which are provided by the exact sequence above), where $U_{\nu\mu}$ and $U_{\nu\mu'}$ are associated to adjoining cubes $E_{\nu\mu}$ and $E_{\nu\mu'}$. Repeating this procedure (recall the number of $E_{\nu\mu}$ is finite), for a suitable $N_\nu \in \mathcal{N}$, we can construct an exact sequence

$$\mathcal{O}_{U_\nu}^{N_\nu} \xrightarrow{\varphi_\nu} \mathcal{F}|_{U_\nu} \longrightarrow 0 \tag{1}$$

on a neighbourhood U_ν of $\overline{\Omega}_\nu$. Hence, we obtain a short exact sequence

$$0 \longrightarrow \ker \varphi_\nu \longrightarrow \mathcal{O}_{U_\nu}^{N_\nu} \xrightarrow{\varphi_\nu} \mathcal{F}|_{U_\nu} \longrightarrow 0 \tag{2}$$

Restricting to Ω_ν, we have a long exact sequence

$$H^q(\Omega_\nu, \mathcal{O}_{\Omega_\nu}^{N_\nu}) \to H^q(\Omega_\nu, \mathcal{F}) \to H^{q+1}(\Omega_\nu, \ker \varphi_\nu) \to H^{q+1}(\Omega_\nu, \mathcal{O}_{\Omega_\nu}^{N_\nu})$$

By Dolbeaut theorem, since Ω_ν is a convex cylinder domain, we get

$$H^q(\Omega_\nu, \mathcal{O}_{\Omega_\nu}^{N_\nu}) = 0 \quad q \geq 1$$

therefore,

$$H^q(\Omega_\nu, \mathcal{F}) \cong H^{q+1}(\Omega_\nu, \ker \varphi_\nu) \quad q \geq 1$$

We have increased the degree of the cohomology group by one, which is the key point in view of the vanishing property of step 1. By (2), $\ker \varphi$ is coherent by Serre theorem 1. As a result, we proved that, given a coherent sheaf $\mathcal{F}$ over a neighbourhood of Ω_ν, we can find a second coherent sheaf $\mathcal{F}_1$ over a possibly smaller neighbourhood of Ω_ν, such that the following is true:

$$H^q(\Omega_\nu, \mathcal{F}) \cong H^{q+1}(\Omega_\nu, \mathcal{F}_1) \quad q \geq 1$$

Iterating this procedure, we get a chain of isomorphisms

$$H^q(\Omega_\nu, \mathcal{F}) \cong H^{q+1}(\Omega_\nu, \mathcal{F}_1) \cong \cdots \cong H^{2^{2n}}(\Omega_\nu, \mathcal{F}_{2^{2n}-q}) = 0$$

($q \geq 1$) where the last equality holds by application of step 1.

STEP 3. Until now, we proved Oka-Cartan fundamental theorem in the case of an open cube and of a sheaf defined over a neighbourhood of its closure. To deal with the general case, we need to glue together local solutions of the δ-equation. We will correct local solutions with appropriate cocycles in order to satysfy the glueing conditions.

Now, our goal is to prove that for any coherent sheaf $\mathcal{F}$ over a convex cylinder domain Ω, $H^q(\Omega, \mathcal{F}) = 0$, for $q \geq 1$.

Again, by the Riemann mapping theorem, we can assume that Ω is an open cube. Let $\{\Omega_\mu\}$ be an exhaustion by relatively compact open cubes, and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ a locally finite open covering of Ω by relatively compact open cubes.

By step 2, since $U_{i_0} \cap \cdots \cap U_{i_q}$ is still an open cube for every $(i_0, \dots, i_q) \in A^{q+1}$, $H^q(U_{i_0} \cap \cdots \cap U_{i_q}, \mathcal{F}) = 0$ for all $q \geq 1$, which shows that $\mathcal{U}$ is a Leray covering. We set $\mathcal{U}_\nu = \{U_\alpha \cap \Omega_\nu\}$, which is a Leray covering of Ω_ν, because $U_\alpha \cap \Omega_\nu$ is still a relatively compact open cube in Ω. Therefore, we obtain the isomorphisms

$$\begin{aligned} H^q(\Omega, \mathcal{F}) &\cong H^q(\mathcal{U}, \mathcal{F}) \quad q \geq 0 \\ 0 = H^q(\Omega_\nu, \mathcal{F}) &\cong H^q(\mathcal{U}_\nu, \mathcal{F}) \quad q \geq 1 \end{aligned} \tag{3}$$

by the property of Leray covering and step 2. In order to finish this step of the proof, we need to show that, for every $f \in Z^q(\mathcal{U}, \mathcal{F})$, we can find a $\tilde{g}_\nu \in C^{q-1}(\mathcal{U}_\nu, \mathcal{F})$ satisfying the following properties:

1. $\delta\,\tilde{g}_\nu = f|_{\Omega_\nu}$;
2. $\tilde{g}_{\nu;i_0,\dots,i_q} = \tilde{g}_{\nu-1;i_0,\dots,i_q}$ for all $(i_0, \dots, i_q) \in A^{q+1}$ such that $U_{i_0} \cap \cdots \cap U_{i_q} \subseteq \Omega_{\nu-1}$.

In fact, by the sheaf axioms, we glue $\{\tilde{g}_\nu\}$ a cochain $\tilde{g} \in C^{q-1}(\mathcal{U}, \mathcal{F})$ such that $\delta\,\tilde{g} = f$. As a consequence, $[f] = 0$ as wanted.

We analyse separately the case $q \geq 2$ and $q = 1$ (in this order).

CASE $q \geq 2$. Condition (3) implies there exist g_ν in $C^{q-1}(\mathcal{U}_\nu, \mathcal{F})$ such that

$$f|_{\Omega_\nu} = \delta\, g_\nu \quad \nu = 1, 2, \dots \tag{4}$$

The following is a standard argument. We correct the defect $g_\nu - g_{\nu+1}$ on Ω_ν by the boundary of a cochain in $C^{q-2}(\mathcal{U}_\nu, \mathcal{F})$, hence providing a variation $\tilde{g}_\nu$ of the sequence g_ν with the properties required. Note that since $q \geq 2$, we have enough room to decrease the degree of the cohomology by two units. We emphasise that this process crashes down in the case $q = 1$, for which a more substantial approximation argument is needed.

Set $\tilde{g}_1 = g_1$. Given an element $\tilde{g}_\nu \in C^{q-1}(\mathcal{U}_\nu, \mathcal{F})$ with $\nu \leq \mu$ satisfying properties 1 and 2 above, our aim is to construct a cochain $\tilde{g}_{\mu+1}$. By (4), it holds that $\delta\,(\tilde{g}_\mu - \tilde{g}_{\mu+1}|_{\Omega_\mu}) = 0$, and there exists $h_{\mu+1} \in C^{q-2}(\mathcal{U}_\mu, \mathcal{F})$ such that

$$\tilde{g}_\mu - \tilde{g}_{\mu+1}|_{\Omega_\mu} = \delta\, h_{\mu+1} \tag{5}$$

We extend $h_{\mu+1}$ to a cochain $\tilde{h}_{\mu+1} \in C^{q-2}(\mathcal{U}, \mathcal{F})$ as follows:

$$\tilde{h}_{\mu+1;i_0,\dots,i_{q-2}} = \begin{cases} h_{\mu+1;i_0,\dots,i_{q-2}} & U_{i_0} \cap \cdots \cap U_{i_{q-2}} \subseteq \Omega_\mu \\ 0 & \text{else} \end{cases} \tag{6}$$

We set

$$\tilde{g}_{\mu+1} = g_{\mu+1} + \delta\,\tilde{h}_{\mu+1}|_{\Omega_{\mu+1}}$$

then, by definition of $\tilde{g}_{\mu+1}$ and (4)

$$\delta\,\tilde{g}_{\mu+1} = \delta\, g_{\mu+1} = f|_{\Omega_{\mu+1}}$$

If $U_{i_0} \cap \cdots \cap U_{i_{q-1}} \subseteq \Omega_\mu$, then by definition of $\tilde{g}_{\mu+1}$, (6) and (5),

$$\begin{aligned} \tilde{g}_{\mu+1;i_0,\dots,i_{q-1}} &= g_{\mu+1;i_0,\dots,i_{q-1}} + \delta\,\tilde{h}_{\mu+1;i_0,\dots,i_{q-1}} \\ &= g_{\mu+1;i_0,\dots,i_{q-1}} + \delta\, h_{\mu+1;i_0,\dots,i_{q-1}} \\ &= \tilde{g}_{\mu;i_0,\dots,i_{q-1}} \end{aligned}$$

which completes the construction of the required sequence.

CASE $q = 1$. The case $q = 1$ requires a different approach that we outline below.

Lemma 13. *Let E be a closed cube in a domain Ω contained in $\mathbb{C}^n$, $\mathcal{F}$ a coherent sheaf over Ω. There exists open neighbourhoods U, U' of E, with $U' \subseteq U$, such that the following propositions are true:*

1. *$\mathcal{F}$ is spanned by finitely many global sections, i.e., for a suitable $N \in \mathcal{N}$, the following sequence is exact:*

$$\mathcal{O}_U^N \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

2. *$\mathcal{F}(U')$ is spanned by finitely many global sections, i.e., for a suitable $N \in \mathcal{N}$, the following sequence is exact:*

$$\mathcal{O}^N(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

Proof. We divide the proof in two parts.

1. This is an immediate consequence of step 2.
2. By the first part, restricting to U', a relatively compact open cube such that $E \Subset U' \Subset U$, for a suitable $N \in \mathcal{N}$ we obtain a short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow \mathcal{O}_{U'}^N \xrightarrow{\varphi} \mathcal{F}|_{U'} \longrightarrow 0$$

Note that $\ker\varphi$ is coherent by Serre theorem 1. Hence we have a long exact sequence

$$H^0(U', \mathcal{O}_{U'}^N) \xrightarrow{\varphi^*} H^0(U', \mathcal{F}) \longrightarrow H^1(U', \ker\varphi) = 0$$

where the last equality follows by step 2. The map φ^* is onto, and this remark concludes the proof.

□

Now we assume that g_ν is a $(q-1)$-cochain on a neighbourhood of $\overline{\Omega_\nu}$, such that

$$f = \delta\, g_\nu$$

is true on a neighbourhood of $\overline{\Omega_\nu}$. Set $\tilde{g}_1 = g_1$. We are going to define inductively $\tilde{g}_\nu$ on a neighbourhood of $\overline{\Omega_\nu}$. Assume that the elements $\tilde{g}_\nu$, $1 \le \nu \le \mu$, are given. By (4), it holds that $\delta\,(g_{\mu+1}|_{\overline{\Omega_\mu}} - \tilde{g}_\mu) = 0$, and there exists $s_{\mu+1}$ on a neighbourhood of $\overline{\Omega_\mu}$ such that

$$g_{\mu+1}|_{\overline{\Omega_\mu}} - \tilde{g}_\mu = s_{\mu+1} \tag{7}$$

By lemma 13, there exist a finite generator system $\{\sigma_{(\nu)j}\}_{j=1}^{M_\nu}$ of $\mathcal{F}$, on a neighbourhood of $\overline{\Omega_\nu}$. Moreover, any section $s_{\mu+1}$ can be expressed as a linear combination of $\sigma_{(\mu+1)j}$ with holomorphic coefficients $a_{(\mu+1)j}$ defined on a neighbourhood of $\overline{\Omega_\mu}$, by commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{O}^{M_{\mu+1}}(U_{\mu+1}) \arrow[r] \arrow[d] & \mathcal{F}(U_{\mu+1}) \arrow[d] & \\
\mathcal{O}^{M_{\mu+1}}(U_\mu) \arrow[r] & \mathcal{F}(U_\mu) \arrow[r] & 0
\end{tikzcd}$$

(here U_ν is an open neighbourhood of $\overline{\Omega_\nu}$). Explicitly, on a neighbourhood of $\overline{\Omega_\mu}$,

$$s_{\mu+1} = \sum_{j=1}^{M_{\mu+1}} a_{(\mu+1)j}\sigma_{(\mu+1)j}$$

By Runge approximation theorem for convex cylinder domains ([Nog16, theorem 1.2.23]), the holomorphic functions $a_{(\mu+1)j}$ defined on a neighbourhood of $\overline{\Omega_\mu}$ are uniformly approximated in $\overline{\Omega_\mu}$ by holomorphic functions $\tilde{a}_{(\mu+1)j}$ defined on a neighbourhood of $\overline{\Omega_{\mu+1}}$, so that

$$\|a_{(\mu+1)j} - \tilde{a}_{(\mu+1)j}\|_{\overline{\Omega_\mu}} = \sup_{z\in\overline{\Omega_\mu}} |a_{(\mu+1)j}(z) - \tilde{a}_{(\mu+1)j}(z)| < \varepsilon$$

where ε will be determined later. We define the section $\tilde{s}_{\mu+1}$ of $\mathcal{F}$ on a neighbourhood of $\overline{\Omega_{\mu+1}}$

$$\tilde{s}_{\mu+1} := \sum_{j=1}^{M_{\mu+1}} \tilde{a}_{(\mu+1)j}\sigma_{(\mu+1)j}$$

and the 0-cochain $\tilde{g}_{\mu+1}$ on a neighbourhood of $\overline{\Omega_{\mu+1}}$

$$\tilde{g}_{\mu+1} := g_{\mu+1} - \tilde{s}_{\mu+1}$$

The following properties hold:

- $\delta\,\tilde{g}_{\mu+1} = \delta\,g_{\mu+1} = f$ on a neighbourhood of $\overline{\Omega_\mu}$;
- $\tilde{g}_{\mu+1} - \tilde{g}_\mu = g_{\mu+1} - \tilde{s}_{\mu+1} - \tilde{g}_\mu = s_{\mu+1} - \tilde{s}_{\mu+1}$ on a neighbourhood of $\overline{\Omega_\mu}$.

We define

$$G_\nu := \tilde{g}_\nu + b_\nu$$

where b_ν is an additive correction of $\tilde{g}_\nu$. We are going to investigate which conditions should be imposed on b_ν such that G_ν satisfies properties 1 and 2. Firstly,

$$f|_{\Omega_\nu} = \delta\,G_\nu = \delta\,\tilde{g}_\nu + \delta\,b_\nu = f|_{\Omega_\nu} + \delta\,b_\nu$$

which implies that b_ν should be a section of Ω_ν. Secondly, G_ν should satisfy

$$G_{\nu+1}|_{\Omega_\nu} = G_\nu$$

Therefore,

$$\begin{aligned} G_{\nu+1}|_{\Omega_\nu} &= \tilde{g}_{\nu+1} + b_{\nu+1} = \tilde{g}_\nu + \tilde{g}_{\nu+1} - \tilde{g}_\nu + b_{\nu+1} \\ &= \tilde{g}_\nu + s_{\nu+1} - \tilde{s}_{\nu+1} + b_{\nu+1} \\ G_\nu &= \tilde{g}_\nu + b_\nu \end{aligned}$$

which implies that b_ν should have the form

$$\begin{aligned} b_\nu &= s_{\nu+1} - \tilde{s}_{\nu+1} + b_{\nu+1} \\ b_\nu &= \sum_{\lambda=\nu}^{\infty} s_{\lambda+1} - \tilde{s}_{\lambda+1} \\ &= \sum_{\lambda=\nu}^{\infty}\sum_{j=1}^{M_{\lambda+1}} (a_{(\lambda+1)j} - \tilde{a}_{(\lambda+1)j})\sigma_{(\lambda+1)j} \end{aligned}$$

By lemma 13, on a neighbourhood of $\overline{\Omega_\nu}$,

$$\sigma_{(\lambda+1)j} = \sum_{k=1}^{M_\nu} \alpha_{(\lambda+1,\nu)jk}\sigma_{(\nu)k}$$

for some holomorphic functions $\alpha_{(\lambda+1,\nu)jk}$ defined on a neighbourhood of $\overline{\Omega_\nu}$.

$$\begin{tikzcd} & \mathcal{F}(U_{\lambda+1}) \arrow[d] & \\ \mathcal{O}^{M_\nu}(U_\nu) \arrow[r] & \mathcal{F}(U_\nu) \arrow[r] & 0 \end{tikzcd}$$

(here U_μ is an open neighbourhood of $\overline{\Omega_\mu}$). Hence,

$$\begin{aligned} b_\nu &= \sum_{\lambda=\nu}^{\infty} \sum_{j=1}^{M_{\lambda+1}} (a_{(\lambda+1)j} - \tilde{a}_{(\lambda+1)j}) \sum_{k=1}^{M_\nu} \alpha_{(\lambda+1,\nu)jk}\sigma_{(\nu)k} \\ &= \sum_{k=1}^{M_\nu} \left(\sum_{\lambda=\nu}^{\infty} \sum_{j=1}^{M_{\lambda+1}} (a_{(\lambda+1)j} - \tilde{a}_{(\lambda+1)j})\alpha_{(\lambda+1,\nu)jk} \right) \sigma_{(\nu)k} \end{aligned}$$

The existence of such b_ν depends on the convergence of the coefficients of $\sigma_{(\nu)k}$. In fact, they converge absolutely and uniformly on Ω_ν, for instance if we choose ε small enough such that

$$\sum_{\lambda=\nu}^{\infty} \sum_{j=1}^{M_{\lambda+1}} \|(a_{(\lambda+1)j} - \tilde{a}_{(\lambda+1)j})\alpha_{(\lambda+1,\nu)jk}\|_{\overline{\Omega_\nu}} < \frac{1}{2^\lambda}$$

for fixed λ and $1 \leq \nu \leq \lambda$, which concludes the proof.

STEP 4.

Definition 14 (Analytic polyhedron)**.** Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Let $\{f_j\}_{j=1}^m$ be a finite collection of holomorphic maps $\Omega \to \mathbb{C}$. The set

$$P := \{z \in \Omega \mid |f_j(z)| < 1, 1 \leq j \leq m\}$$

is called a $\mathcal{O}(\Omega)$-*semianalytic polyhedron*. A union of finitely many relatively compact connected components of P is called a $\mathcal{O}(\Omega)$-*analytic polyhedron*. The functions $\{f_j\}_{j=1}^m$ will be called the *defining functions* of P.

Analogously, we can define a $\mathcal{O}(M)$-analytic polyhedron when M is a complex manifold.

We need to prove that for any coherent sheaf $\mathcal{F}$ over an analytic polyhedron P in a domain Ω (semianalytic polihedron contained in a convex cylinder domain), $H^q(P, \mathcal{F}) = 0$, for $q \geq 1$.

Let $\{f_j\}_{j=1}^m$ be the analytic functions defining P. Since P is bounded (relatively compact in $\mathbb{C}^n$), there exists a polydisk Δ containing P.[2] The *Oka map* is the holomorphic embedding

$$\begin{aligned} \iota_P \colon P &\to \Delta \times D(0,1)^m \\ z &\mapsto (z, f_1(z), \ldots, f_m(z)) \end{aligned}$$

The image $\iota(P)$ is a closed complex submanifold of the polydisk $\Delta \times D(0,1)^m$.

Indeed, as $|f_j| = 1$ on the boundary points of P for some $j \in \{1, \ldots, m\}$, $\iota(\partial P) \subseteq \partial(\Delta \times D(0,1)^m)$, moreover since ι extends continuously to a map $\overline{P} \to \overline{\Delta \times D(0,1)^m}$, $\overline{P}$ is sent to a compact (hence closed) set. Now, taking the intersection $\iota(\overline{P}) \cap \Delta \times D(0,1)^m = \iota(P)$, we see that $\iota(P)$ is closed in $\Delta \times D(0,1)^m$.

2 Here, we can drop the relative compactness assumption, just requiring that P belongs to a convex cylinder domain. This allows us to extend the result to a semianalytic polihedron contained in a convex cylinder domain.

Identifying P with its image via ι, we are now able to take the simple extension sheaf $\widehat{\mathcal{F}}$ over $\Delta \times D(0,1)^m$, which is still coherent by 5. Applying step 3 to the convex cylinder domain $\Delta \times D(0,1)^m$, we get

$$H^q(\Delta \times D(0,1)^m, \widehat{\mathcal{F}}) = 0 \quad q \geq 1$$

By 6, $H^q(P, \mathcal{F}) \cong H^q(\Delta \times D(0,1)^m, \widehat{\mathcal{F}})$. This terminates the fourth step.

STEP 5. We begin with a technical lemma.

Lemma 15. *A holomorphically convex domain Ω admits an exhaustion by $\mathcal{O}(\Omega)$-analytic polihedrons.*

Proof. Since Ω is a connected domain in $\mathbb{C}^n$, hence second countable and locally compact, it admits an exhaustion $\{V_\nu\}$ by compact connected subsets. By definition of holomorphically convex domain, the holomorphic hull $\widehat{\overline{V_1}}$ of $\overline{V_1}$ in Ω is compact. We choose a relatively compact open neighbourhood W of $\overline{V_1}$. We have a chain of inclusions

$$\overline{V_1} \Subset \widehat{\overline{V_1}} \Subset W \Subset \Omega$$

By definition of holomorphic hull, for each $a \in \partial W$, since $a \notin \widehat{\overline{V_1}}$, there exists a function $f \in \mathcal{O}(\Omega)$ such that

$$\sup_{\widehat{\overline{V_1}}} |f| < |f(a)|$$

For a certain $\theta \in \mathbb{R}$,

$$\sup_{\widehat{\overline{V_1}}} |f| < \theta < |f(a)|$$

By continuity, in a neighbourhood U_a of a,

$$\sup_{\widehat{\overline{V_1}}} |f| < \theta < |f(z)| \quad \forall z \in U_a$$

Eventually rescaling f, we can suppose

$$\sup_{\widehat{\overline{V_1}}} |f| < 1 < |f(z)| \quad \forall z \in U_a$$

Since the boundary of W is compact (the boundary of W can be covered by a finite number of U_a), there exists a finite number of $f_j \in \mathcal{O}(\Omega)$, $j = 1, \dots, m$, such that

$$\widehat{\overline{V_1}} \subseteq P := \{z \in \Omega \mid |f_j(z)| < 1, 1 \leq j \leq m\}$$

Therefore, the connected component P_1 of P containing $\overline{V_1}$, is an analytic polyhedron such that

$$V_1 \Subset P_1 \Subset W$$

Iterating the procedure for V_{ν_2}, with ν_2 large enough such that $\overline{P}_1 \cup \overline{V}_2 \subseteq V_{\nu_2}$, we obtain an exhaustion of Ω by analytic polyhedra P_ν. □

We state our claim: for any coherent sheaf $\mathcal{F}$ over a holomorphically convex domain Ω, $H^q(\Omega, \mathcal{F}) = 0$, for $q \geq 1$.

Basically, we are going over the argument proposed in step 3, with a few changes. Let $\{P_\mu\}$ be an exhaustion by analytic polyhedra of Ω, and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ a locally finite open covering of Ω by relatively compact convex cylinder domains. By step 2, since $U_{i_0} \cap \cdots \cap U_{i_q}$ is still a convex cylinder domaim for every $(i_0, \dots, i_q) \in A^{q+1}$, $H^q(U_{i_0} \cap \cdots \cap U_{i_q}, \mathcal{F}) = 0$ for all $q \geq 1$, which shows that $\mathcal{U}$ is a Leray covering.

We set $\mathcal{U}_\nu = \{U_\alpha \cap P_\nu\}$, which is a Leray covering of P_ν. Indeed, if $U_\alpha \subseteq P_\nu$, then $U_\alpha \cap P_\nu = U_\alpha$ is a convex cylinder domain, otherwise that intersection is a semianalytic polyhedron in a convex cylinder domain. By step 2 and step 4, $\mathcal{U}$ is a Leray covering as claimed. Therefore, we obtain the isomorphisms

$$H^q(\Omega, \mathcal{F}) \cong H^q(\mathcal{U}, \mathcal{F}) \quad q \geq 0$$
$$0 = H^q(P_\nu, \mathcal{F}) \cong H^q(\mathcal{U}_\nu, \mathcal{F}) \quad q \geq 1$$

by the property of Leray covering and step 4. In order to finish this step of the proof, we need to show that, for every $f \in Z^q(\mathcal{U}, \mathcal{F})$, we can find a $\tilde{g}_\nu \in C^{q-1}(\mathcal{U}_\nu, \mathcal{F})$ satisfying the following properties:

- $\delta \tilde{g}_\nu = f|_{P_\nu}$;
- $\tilde{g}_{\nu;i_0,\ldots,i_q} = \tilde{g}_{\nu-1;i_0,\ldots,i_q}$ for all $(i_0, \ldots, i_q) \in A^{q+1}$ such that $U_{i_0} \cap \cdots \cap U_{i_q} \subseteq P_{\nu-1}$.

In fact, by the sheaf axioms, we glue $\{\tilde{g}_\nu\}$ a cochain $\tilde{g} \in C^{q-1}(\mathcal{U}, \mathcal{F})$ such that $\delta \tilde{g} = f$. As a consequence, $[f] = 0$ as wanted.

Again, we have two cases: $q \geq 2$ and $q = 1$. The first one follows easily by replacing Ω_ν by P_ν in the $(q \geq 2)$-case given in step 3. The case $q = 1$ requires a different kind of approximation argument that we outline below.

Lemma 16. *Let P be an analytic polyhedron in a domain Ω contained in $\mathbb{C}^n$, let $\mathcal{F}$ a coherent sheaf over Ω and U an open neighbourhood of $\overline{P}$. The following propositions are true:*

1. *$\mathcal{F}$ is spanned by finitely many global sections, i.e., for a suitable $N \in \mathcal{N}$, the following sequence is exact:*

$$\mathcal{O}_U^N \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

2. *$\mathcal{F}(P)$ is spanned by finitely many global sections, i.e., for a suitable $N \in \mathcal{N}$, the following sequence is exact:*

$$\mathcal{O}^N(P) \longrightarrow \mathcal{F}(P) \longrightarrow 0$$

Proof. We divide the proof in two parts.

1. Let $\{f_j\}_{j=1}^m$ be the analytic functions defining P. Recall that P is a finite union of relatively compact connected components of

$$\{z \in \Omega \mid |f_j(z)| < 1, 1 \leq j \leq m\}$$

For $\varepsilon > 0$, we define $\tilde{P}$, a small perturbation of P, i.e., the union of connected components of

$$\{z \in \Omega \mid (1-\varepsilon)|f_j(z)| < 1, 1 \leq j \leq m\}$$

which contains a point of P. We can choose ε small enough such that

$$P \Subset \tilde{P} \Subset \Omega$$

We allow the following identifications:

- $\mathcal{F}$ is identified with $\mathcal{F}|_{\tilde{P}}$;
- $\tilde{P}$ is identified with its image via the Oka map $\iota_{\tilde{P}}$,

$$\begin{aligned} \iota_{\tilde{P}} \colon \tilde{P} &\to \Delta \times D(0,1)^m \\ z &\mapsto (z, (1-\varepsilon)f_1(z), \ldots, (1-\varepsilon)f_m(z)) \end{aligned}$$

with $\tilde{P}$ contained in the polydisk Δ;

- $\Delta \times D(0,1)^m$ is identified with an open cube R via the Riemann mapping theorem.

The simple extension $\widehat{\mathcal{F}}$ of $\mathcal{F}$ over R is coherent by 5 (note that $\tilde{P}$ is a closed complex submanifold of R, hence $\widehat{\mathcal{F}}$ is well defined). Since P is relatively compact in R, there exists an open cube E such that $P \subseteq E \subseteq R$. By step 2, for a suitable $N \in \mathcal{N}$ we obtain the following exact sequence:

$$\mathcal{O}_V^N \longrightarrow \widehat{\mathcal{F}}|_V \longrightarrow 0$$

on a neighbourhood V of E. Restricting to $U := V \cap \tilde{P}$, we obtain the sequence stated in the claim.

2. This proof proceed as in lemma 13, replacing the open neighbourhood U' by the analytic polyhedron P.

□

We need one more lemma.

Lemma 17 (Runge-Oka approximation)**.** *Let P be an analytic polyhedron in a domain Ω contained in $\mathbb{C}^n$. Every holomorphic function defined over P is arbitrarily approximated uniformly on compact subsets of P by elements in $\mathcal{O}(\Omega)$.*

Proof. Let the Oka map of P be

$$\begin{aligned} \iota_P \colon P &\to \Delta \times D(0,1)^m =: \Delta' \\ z &\mapsto (z, f_1(z), \ldots, f_m(z)) \end{aligned}$$

where $\{f_j\}_{j=1}^m$ are the analytic functions defining P. As in definition 3, we are given the short exact sequence

$$0 \longrightarrow \mathcal{I}_P \longrightarrow \mathcal{O}_{\Delta'} \longrightarrow \widehat{\mathcal{O}_P} \longrightarrow 0$$

where as usual we identify P with its image $\iota_P(P)$. The following long exact sequence is exact

$$H^0(\Delta', \mathcal{O}_{\Delta'}) \longrightarrow H^0(\Delta', \widehat{\mathcal{O}_P}) \cong H^0(P, \mathcal{O}_P) \longrightarrow H^1(\Delta', \mathcal{I}_P) = 0$$

where the last equality holds by coherence of $\mathcal{I}_P$ and step 3. More explicitly,

$$\mathcal{O}(\Delta') \longrightarrow \mathcal{O}(P) \longrightarrow 0$$

It follows that, for any $f \in \mathcal{O}(P)$, there exists a holomorphic function $F \in \mathcal{O}(\Delta')$ such that $F|_P = f$. We expand $F(z,w)$ (where $(z,w) \in \Delta \times D(0,1)^m$) in power series,

$$F(z,w) = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha w^\beta$$

For every compact subset K in P, and $\varepsilon > 0$, it exists $N \in \mathbb{N}$ such that

$$|F(z,w) - \sum_{|\alpha|,|\beta|<N} a_{\alpha\beta} z^\alpha w^\beta| < \varepsilon \quad \forall (z,w) \in K \subseteq \Delta'$$

Replacing w with $h = (f_1, \ldots, f_m)$, we get

$$|f(z) - \sum_{|\alpha|,|\beta|<N} a_{\alpha\beta} z^\alpha h^\beta(z)| < \varepsilon \quad \forall (z,w) \in K \subseteq P \subseteq \Omega$$

Since $\sum_{|\alpha|,|\beta|<N} a_{\alpha\beta} z^\alpha h^\beta(z) \in \mathcal{O}(\Omega)$, the result holds. □

At this point, the proof of the $(q = 1)$-case proceeds as in step 3, replacing open cubes Ω_ν with analytic polyhedra P_ν and Runge approximation with Runge-Oka approximation.

This was the final step, so the proof is complete.

3 STEIN MANIFOLDS

We begin with a definition.

Definition 18. Let M be a complex manifold of dimension n. M is a *Stein manifold* if

- (holomorphic separability) for any two points $x, y \in M$ with $x \neq y$, there exists $f \in \mathcal{O}(M)$ such that $f(x) \neq f(y)$;
- (globally defined holomorphic local charts) M admits a complex atlas of the form $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$, where $\varphi_\alpha \in \mathcal{O}(M)^n$ for all $\alpha \in A$;
- (holomorphic convexity) M is holomorphically convex.

Any holomorphically convex domain of $\mathbb{C}^n$ is a Stein manifold. On the contrary, although compact complex manifolds are trivially holomorphically convex, globally defined holomorphic functions don't separate points, since by the maximum principle they are constant sections. Hence, compact manifolds are not Stein.

Lemma 19. *Let P be a relatively compact open subset of a Stein manifold M. Then there exists an injective immersion of P in a polydisk of dimension large enough.*

Proof. Let M be endowed with a complex atlas made up of globally defined holomorphic local charts. By compactness of $\overline{P}$, we can extract from the atlas a finite open covering $\{U_\alpha\}_{\alpha=1}^{l}$ of $\overline{P}$, with associated coordinate charts $\{\varphi_\alpha\}_{\alpha=1}^{l}$. Fix $x \in \overline{P}$. By holomorphic separability of M, for all $y \in \overline{P} \setminus \{x\}$, there exists a function $f_{xy} \in \mathcal{O}(M)$, such that $f_{xy}(x) \neq f_{xy}(y)$. By continuity of f_{xy}, there exist open neighbourhoods A_{xy} of x and B_{xy} of y, such that $f_{xy}(A_{xy}) \cap f_{xy}(B_{xy}) = \emptyset$.

We can suppose $A_{xy} \subseteq U_\alpha$ and $B_{xy} \subseteq U_\beta$ for some $\alpha, \beta \in \{1, \ldots, l\}$. The collection $\{A_{xy} \cup B_{xy}\}_{y \in \overline{P}}$ is an open covering of $\overline{P}$. Again, we extract a finite covering $\{A_{xy_\mu} \cup B_{xy_\mu}\}_{\mu=1}^{m(x)}$. We define

$$A_x = \bigcap_{\mu=1}^{m(x)} A_{xy_\mu}$$

The family $\{A_x\}_{x \in \overline{P}}$ is an open covering of $\overline{P}$. We extract a finite covering $\{A_{x\nu}\}_{\nu=1}^{n}$. We define the map $f \colon \overline{P} \to \mathbb{C}^N$ by

$$f(z) = (\varphi_1(z), \ldots, \varphi_l(z), f_{x_1 y_1}(z), \ldots, f_{x_1 y_{m(x_1)}}, \ldots, f_{x_n y_1}, \ldots, f_{x_n y_{m(x_n)}})$$

Claim: the map f is injective.

Suppose that $z, w \in \overline{P}$, $z \neq w$, such that $f(z) = f(w)$. Since $\{A_{x\nu}\}_{\nu=1}^{n}$ is a covering of $\overline{P}$, $z \in A_{x_\nu}$ for some $\nu \in \{1, \ldots, n\}$. Recall that

$$A_{x_\nu} = \bigcap_{\mu=1}^{m(x_\nu)} A_{x_\nu y_\mu}$$

If w belongs to $A_{x_\nu y_\mu}$ for some $\mu \in \{1, \ldots, m(x_\nu)\}$, then, since also $z \in A_{x_\nu y_\mu}$, w, z are in the same coordinate open U_α for some $\alpha \in \{1, \ldots, l\}$. Hence $\varphi_\alpha(z) \neq \varphi_\alpha(w)$, which is a contradiction. As a consequence, since $\{A_{x_\nu y_\mu} \cup B_{x_\nu y_\mu}\}_{\mu=1}^{m(x_\nu)}$ is an open covering of $\overline{P}$, w belongs to $B_{x_\nu y_\mu}$ for some $\mu \in \{1, \ldots, m(x_\nu)\}$. Hence $f_{x_\nu y_\mu}(z) \neq f_{x_\nu y_\mu}(w)$, which is a contradiction. Our claim is proved.

By continuity of f, the image of P is relatively compact in $\mathbb{C}^N$, therefore contained in a polydisk Δ of suitable radius. Since the functions φ_α are local charts of P, $f|_P$ is an injective immersion. $\square$

Now consider an analytic polyhedron P in M, the Oka map of P, defined as

$$\begin{aligned} \iota_P \colon P &\to \Delta \times D(0,1)^m \\ z &\mapsto (f(z), f_1(z), \ldots, f_m(z)) \end{aligned}$$

where f is the injective immersion of lemma 19 and $\{f_j\}_{j=1}^m$ are the analytic functions defining P. As in step 4, it is a closed immersion, hence an embedding.

Remark 20. A stronger result, which will not be proved in these notes, is that every Stein manifold is embeddable in $\mathbb{C}^N$ for N large enough.

Theorem 21 (Oka-Cartan fundamental theorem on Stein manifolds)**.** *Let M be a Stein manifold, and let $\mathcal{F}$ be a coherent sheaf over M. Then we have*

$$H^q(M, \mathcal{F}) = 0 \quad q \geq 1$$

Proof. There is a locally finite open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ such that every U_α is biholomorphic to a holomorphically convex domain. Indeed, M is locally biholomorphic to an open in $\mathbb{C}^n$ ($\dim M = n$), and the inverse image of any holomorphic convex domain (e.g., a polydisk) via local charts is still a holomorphic convex domain by 11.

Since a connected component of a finite intersection of U_α is a holomorphically convex domain (by 11), $\mathcal{U}$ is a Leray covering with respect to any coherent sheaf. Note that the exhaustion argument proposed in Lemma 15 relies exclusively on the holomorphic convexity of Ω, hence it still holds for M.

By the same token, the results shown in step 4 and 5 for analytic polyhedra in a holomorphic convex domain, remain true for analytic polyhedra in a Stein manifold. Indeed, also in this setting, the Oka map and the holomorphically convex Leray covering of M are given. $\square$

Some corollaries of the Oka-Cartan fundamental theorem are shown below.

Corollary 22 (Analytic de Rham theorem)**.** *Let M be a Stein manifold of dimension n. Then it holds that*

$$H^q(M, \mathbb{C}) \cong H^q_{AdR}(M, \mathbb{C}) \quad q \geq 0$$

In particular, $H^q(M, \mathbb{C}) = 0$ when $q > n$.

Proof. In the case of Stein manifolds, the following resolution of the constant sheaf $\mathbb{C}$ is acyclic:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_M = \mathcal{O}_M^{(0)} \longrightarrow \mathcal{O}_M^{(1)} \longrightarrow \dots \longrightarrow \mathcal{O}_M^{(n)} \longrightarrow 0$$

Indeed, $\mathcal{O}_M^{(\mu)}$ are locally free sheaves over $\mathcal{O}_M$, hence coherent. By Oka-Cartan fundamental theorem, $H^q(M, \mathcal{O}_M^{(\mu)}) = 0$ for $q \geq 1$ and $\mu \geq 0$. The claim follows from the abstract de Rham theorem. $\square$

Corollary 23 (Cartan theorem A)**.** *Let M be a Stein manifold and $\mathcal{F}$ a coherent sheaf over M. Then $\mathcal{F}$ is spanned by finitely many global sections, i.e., for a suitable $N \in \mathcal{N}$, the following sequence is exact:*

$$\mathcal{O}_M^N \longrightarrow \mathcal{F} \longrightarrow 0$$

Proof. Let X be a closed complex submanifold of M, the following sequence is exact:

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M / \mathcal{I}_X \longrightarrow 0$$

Let $X = \{a\}$. Tensoring with the $\mathcal{O}_M$-coherent sheaf $\mathcal{O}_M$, we obtain the following exact sequence of coherent sheaves by 2:

$$\mathcal{I}_{\{a\}} \otimes_{\mathcal{O}_M} \mathcal{F} \longrightarrow \mathcal{O}_M \otimes_{\mathcal{O}_M} \mathcal{F} \cong \mathcal{F} \longrightarrow \mathcal{O}_M / \mathcal{I}_{\{a\}} \otimes_{\mathcal{O}_M} \mathcal{F} \longrightarrow 0$$

Let $\mathcal{K}$ denote the kernel of the map $\mathcal{F} \to \mathcal{O}_M / \mathcal{I}_{\{a\}} \otimes_{\mathcal{O}_M} \mathcal{F} \cong \mathcal{F} / \mathcal{I}_{\{a\}} \mathcal{F}$. Moreover, observe that if $x \neq a$, since $\mathcal{I}_{\{a\},x} = \mathcal{O}_{M,x}$,

$$\left(\mathcal{F} / \mathcal{I}_{\{a\}} \mathcal{F} \right)_x = 0$$

and if $x = a$,

$$\left(\mathcal{F}/\mathcal{I}_{\{a\}}\mathcal{F}\right)_a = \mathcal{F}_a/\mathfrak{m}_a\mathcal{F}_a$$

where $\mathfrak{m}_a$ is the maximal ideal of the local ring $\mathcal{O}_{M,a}$. Hence $\mathcal{F}/\mathcal{I}_{\{a\}}\mathcal{F}$ is the skyscraper sheaf $\widehat{\mathcal{F}_a/\mathfrak{m}_a\mathcal{F}_a}$ centered in a. By Serre theorem 1, K is coherent, and the following short exact sequence holds:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{I}_{\{a\}}\mathcal{F} \longrightarrow 0$$

We obtain the long exact sequence

$$H^0(M, \mathcal{F}) \longrightarrow H^0(M, \widehat{\mathcal{F}_a/\mathfrak{m}_a\mathcal{F}_a}) \longrightarrow H^1(M, \mathcal{K}) = 0$$

where the last equality holds by the Oka-Cartan fundamental theorem. More explicitly,

$$\mathcal{F}(M) \longrightarrow \mathcal{F}_a/\mathfrak{m}_a\mathcal{F}_a \longrightarrow 0$$

The surjectivity of this map implies there exist finitely many global sections of $\mathcal{F}(M)$ such that their image span $\mathcal{F}_a/\mathfrak{m}_a\mathcal{F}_a$ (which is a finitely generated $\mathcal{O}_{M,a}$-module by coherence of $\mathcal{F}$). By Nakayama lemma, the same sections are generators of $\mathcal{F}_a$. □

Remark 24. In the literature, the previous result is commonly known as Cartan theorem A. On the other hand, the Oka-Cartan fundamental theorem is usually called Cartan theorem B. Any locally finite $\mathcal{O}_M$-module (e.g., a coherent sheaf) is spanned by local sections ([Nog16, Prop 2.4.6, Point-Local Generation]). For coherent sheaves over Stein manifolds, Cartan theorem A proves "global-point generation".

REFERENCES

[Dem12] J.-P. Demailly. *Complex analytic and differential geometry*. 2012. URL: www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf (cit. on pp. 2, 4).

[GH78] P. Griffits and J. Harris. *Principles of Algebraic Geometry*. Wiley-Interscience Publication, 1978 (cit. on p. 5).

[Nog16] J. Noguchi. *Analytic Function Theory of Several Variables*. Springer, 2016 (cit. on pp. 1, 2, 4, 6, 9, 16).